Derivatives of Inverse Functions
Lecture 4

First recall that inverse function is analogous to the undo process;

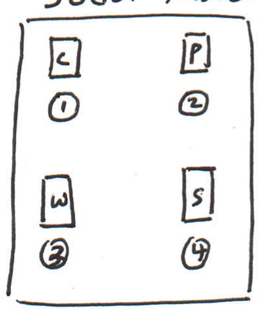
Ex. $f(x) = 3x + 6$ (input multiplied by 3 and then added to 6). How to undo?

$$f(x) - 6 = 3x \implies \frac{f(x)-6}{3} = x.$$

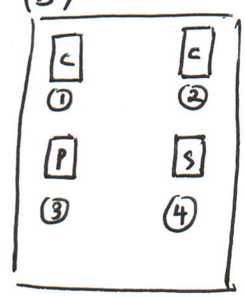
To undo, subtract 6 and divide by 3. Thus the undo function is $f^{-1}(x) = \frac{x-6}{3}$ (subtract 6 and divide by 3).

Not every function has an undo.

Ex. (a) Soda Machine



(b) Soda Machine

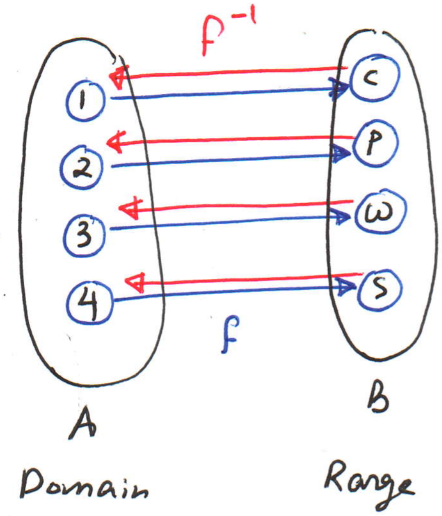


$f: \{1,2,3,4\} \longmapsto \{c, P, w, s\}$. $g: \{1,2,3,4\} \longmapsto \{c, P, s\}$

Soda machines (a) and (b) may be thought as functions that transform numbers to soda cans. Which of the two functions has an "undo" process?

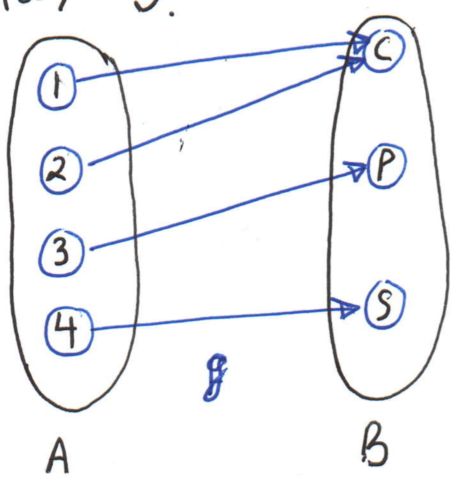
Solution:

(a)



Notice that f is one-to-one (injective) from A to B. and f transforms numbers into soda cans. The undo function $f^{-1}: B \rightarrow A$ must transform soda cans back to numbers. For instance $f^{-1}(w) = 3$.

(b)



Notice that g isn't one-to-one (not injective) from A to B. We cannot define $g^{-1}(c)$ because we cannot figure out if the input that produced c was 1 or 2.

<u>Def:</u> If a function $f: A \to B$ is one-to-one and $B = f(A)$, we say that $f^{-1}: B \to A$ is the inverse function of f if

$$f^{-1} \circ f(x) = x \quad \text{and} \quad f \circ f^{-1}(x) = x$$

In other words f and f^{-1} undo each other.

<u>Remark:</u> $f^{-1}(x) \neq \dfrac{1}{f(x)}$ for example, if $f(x) = 3x+6$

$$f^{-1}(x) = \frac{x-6}{3} \neq \frac{1}{3x+6} \;!$$

The reason for the notation is as follows:
Just as with multiplication, we can create a notation for function composition. Let $f(x) = x+2$

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$$

$f^0(x) \equiv$ apply f to x 0 times

$f^0(x) = x$

$f^1(x) \equiv$ apply f to x 1 time

$f^1(x) = f(x) = x+2$

$f^2(x) \equiv$ apply f twice

$$f^2(x) = f \circ f(x) = f(f(x)) = f(x+2) = (x+2)+2$$
$$= x + 2 \cdot 2$$

$$f^3(x) = x + 3 \cdot 2$$

$$f^5(x) = x + 5 \cdot 2.$$

In general $f^n(x)$ means "apply f n times".

In the case where $f(x) = 2+x$, $f^n(x) = x + n \cdot 2$.

Notice that $f^n \circ f^m(x) = f^{n+m}(x)$ so the analogy of function composition and exponents is strong:

The natural notation for inverse functions (undo) is therefore f^{-1}, because $f^{-1} \circ f^1 = f^{-1+1} = id$.

Ex. Let $f(x) = 3x$

Compute (a) $f^4(x)$ (b) $f^{-1}(x)$ (c) $f^{-3}(x)$.

Solution:

(a) Observe that $f^2(x) = f(f(x)) = f(3x) = 3(3x) = 3^2 \cdot x$

Hence $f^4(x) = f^2(f^2(x)) = 3^2(3^2 x) = 3^4 \cdot x$.

In general $f^n(x) = 3^n \cdot x$.

(b) $f^{-1}(x)$ must satisfy $f(f^{-1}(x)) = x$. Thus

$f(f^{-1}(x)) = 3f^{-1}(x) = x$ or $f^{-1}(x) = \frac{x}{3}$.

(c) $f^{-3}(x) = f^{-2}(f^{-1}(x)) = f^{-2}(\frac{x}{3}) = f^{-1}(f^{-1}(\frac{x}{3})) =$

$= f^{-1}(\frac{x}{3^2}) = \frac{x}{3^3}$.

<u>Ex.</u> Compute f^{-1}

(a) $f(x) = 5x + 8$

(b) $f(x) = \frac{1}{x}$

(c) $f(x) = \frac{5}{x-1}$

(d) $f(x) = x^3$

(e) $f(x) = \frac{x-1}{x+1}$

(f) $f(x) = \frac{x^5 - 2}{x^5 + 7}$

<u>Solution:</u>

(a) $f(\underbrace{f^{-1}(x)}_{y}) = 5[\underbrace{f^{-1}(x)}_{y}] + 8 = x$

Writing $f^{-1}(x)$ is cumbersome so we abbreviate to $y = f^{-1}(x)$.

$f(y) = 5y + 8 = x$ so $y = \frac{x-8}{5}$ Hence $\boxed{f^{-1}(x) = \frac{x-8}{5}}$

(b) Let $y = f^{-1}(x)$ then $f(y) = f(f^{-1}(x)) = x$

or $\frac{1}{y} = x$ Hence $y = \frac{1}{x}$ and $\boxed{f^{-1}(x) = \frac{1}{x} = f(x)}$

f is its own inverse!

(c) Set $y = f^{-1}(x)$. Then $f(y) = x$ so

$x = f(y) = \frac{5}{y-1}$; $\frac{x}{5} = \frac{1}{y-1}$, $\frac{5}{x} = y - 1$;

$y = \frac{5}{x} + 1 = \frac{5+x}{x}$ so $\boxed{f^{-1}(x) = \frac{x+5}{x}}$.

(d) set $y = f^{-1}(x)$ $\quad f(y) = y^3 = x$. so $y = \sqrt[3]{x}$

or $\boxed{f^{-1}(x) = \sqrt[3]{x}}$

(e) set $y = f^{-1}(x)$. $\quad f(y) = \dfrac{y-1}{y+1} = x$ so

$$\frac{(y+1)-2}{(y+1)} = 1 - \frac{2}{y+1} = x \Rightarrow 1 - x = \frac{2}{y+1}$$

$$\Rightarrow y+1 = \frac{2}{1-x} \quad \Rightarrow y = \frac{2}{1-x} - 1 = \frac{2-(1-x)}{1-x}$$

$$= \frac{1+x}{1-x} \quad \text{so} \quad \boxed{f^{-1}(x) = \frac{1+x}{1-x}}$$

(f) set $y = f^{-1}(x)$ $\quad f(y) = \dfrac{y^5 - 2}{y^5 - 7} = x$

$$\frac{(y^5-7)+5}{(y^5-7)} = 1 + \frac{5}{y^5-7} = x$$

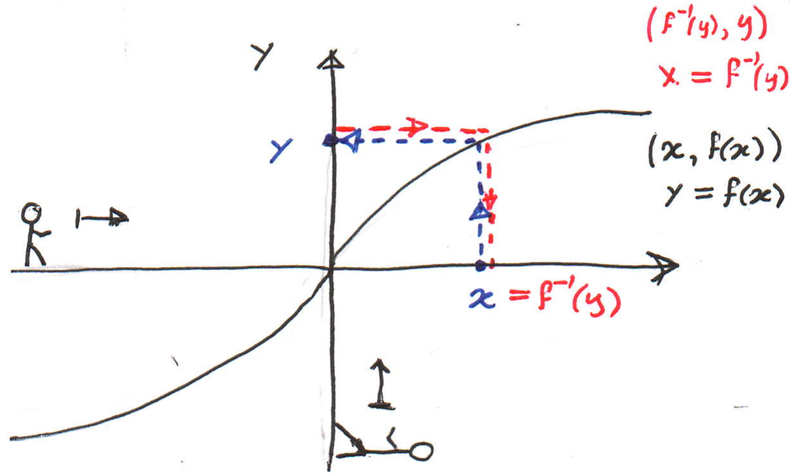
$$\frac{y^5-7}{5} = \frac{1}{x-1} \qquad y^5 - 7 = \frac{5}{x-1}, \quad y^5 = \frac{5}{x-1} + 7$$

$$y^5 = \frac{5+7(x-1)}{x-1} = \frac{7x-2}{x-1} \quad \text{or} \quad y = \sqrt[5]{\frac{7x-2}{x-1}}$$

$$\boxed{f^{-1}(x) = \sqrt[5]{\frac{7x-2}{x-1}}}$$

Graphs of inverse functions

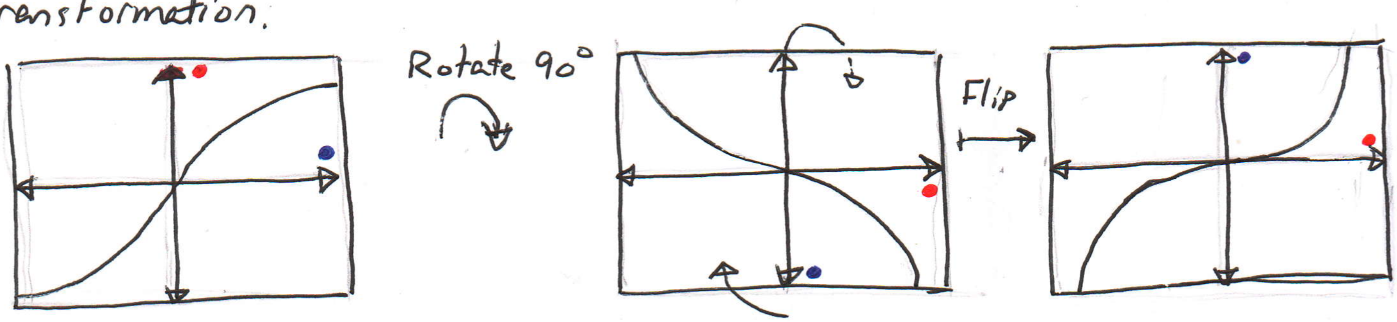
Notice that if the graph $y = f(x)$ passes the horizontal line test, then f is invertible. When you're looking at the graph $y = f(x)$ you are staring at the graph of the inverse as a bonus.



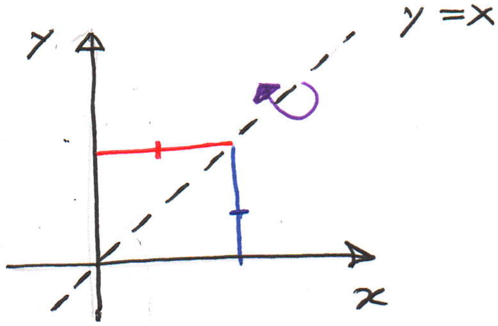
If you're walking on the x-axis and look up, you're seeing the graph $y = f(x)$.

If you're walking on the y-axis and look up (i.e. on x-axis) you're seeing the graph $x = f^{-1}(y)$.

Since we're accustomed to present the input variable on the horizontal axis just perform the following transformation.



We can also observe that the y and x axis can be swapped in one swift move by reflection in the line $y = x$

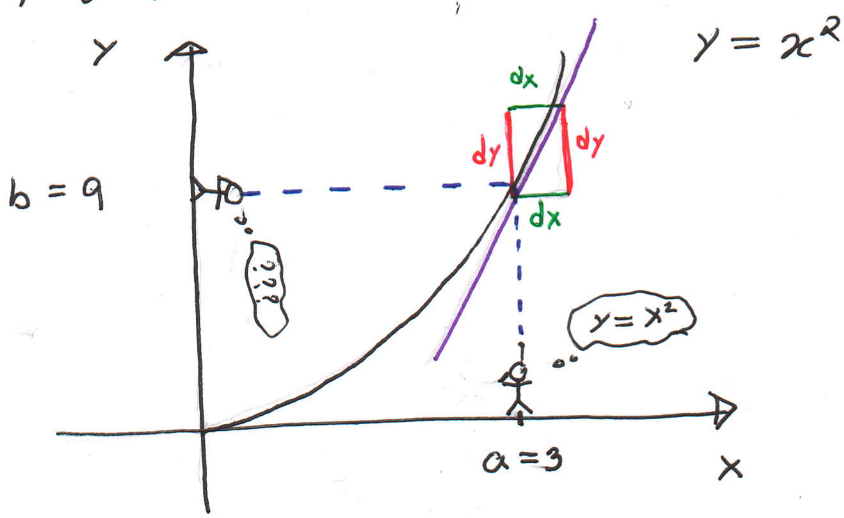


From the geometric perspective f and f^{-1} are the same functions.

<u>Ex.</u> Astronauts a and b are doing a space walk

(i) What function does b see if astraunaut a exclaims: I see the square function!

(ii) What derivatives does each see?



<u>Solution:</u>

(i) Since a sees $y = x^2$, b sees $x = \sqrt{y}$

(ii) a measures slope $\frac{dy}{dx} = \frac{d}{dx}(x^2)\Big|_{x=3} = 2 \cdot 3 = 6$

But b is watching the same line! He is just reading the slope as $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{6}$, because for him y-axis is horizontal and x-axis is vertical.

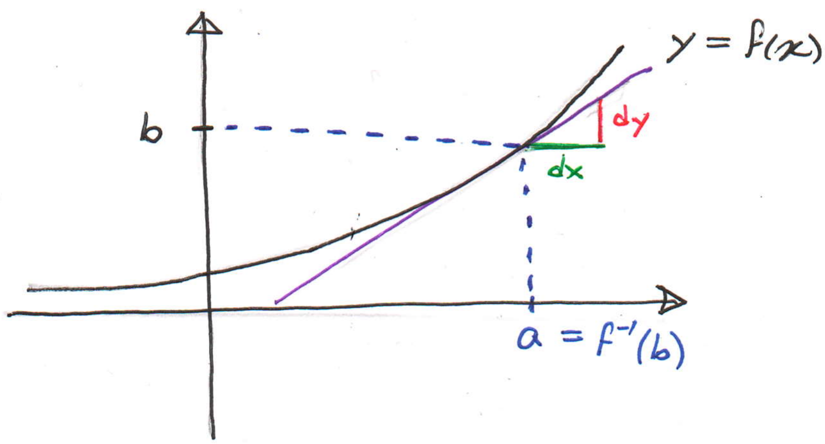
This is the situation in general.

<u>Thm:</u> Let f be a one-to-one differentiable function. Then f^{-1} is invertible with derivative $(f^{-1}(b))' =$

$$= \frac{1}{f'(f^{-1}(b))}.$$

This is defined whenever $f'(f^{-1}(b)) \neq 0$.

<u>Proof:</u>



$(f^{-1}(b))' \equiv$ slope of tangent line at $(f^{-1}(b), b) = (a, f(a))$

Clearly this slope is $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}\Big|_{x=a} = \frac{1}{f'(a)}$

$$= \frac{1}{f'(f(b))}.$$

Can you prove this thm by applying the definition of derivative?

Ex. Use inverse functions to establish the derivative formula.

(a) $\frac{d}{dx}\left(\sqrt{x}\right)$ (d) $\frac{d}{dx}\left(\ln x\right)$

(b) $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ (e) $\frac{d}{dx}\left(\log_3 x\right)$.

(c) $\frac{d}{dx}\left(\sqrt[n]{x}\right)$ (f) $\frac{d}{dx}\left(\log_a x\right)$

Solution:

(a) Set $f^{-1}(x) = \sqrt{x}$ then $f(x) = x^2$

Then $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2(f^{-1}(x))} = \frac{1}{2\sqrt{x}}$

(b) set $f^{-1}(x) = \sqrt[3]{x}$ then $f(x) = x^3$

Then $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{3(\sqrt[3]{x})^2} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-\frac{2}{3}}$

(c) Set $f^{-1}(x) = x^{1/n}$, $f(x) = x^n$

Then $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n}\frac{1}{x^{\frac{n-1}{n}}}$

$= \frac{1}{n} x^{\frac{1}{n}-1}$!

(d) set $f^{-1}(x) = \ln x$ then $f(x) = e^x$

Then $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$

(e) Set $f^{-1}(x) = \log_3 x$ then $f(x) = 3^x$.

Notice that $f'(x) = 3^x \ln 3$ so $\frac{d}{dx} f^{-1}(x) = \frac{1}{3^{\log_3 x}\ln 3} = \frac{1}{x\ln 3}$.

(f) set $f^{-1}(x) = \log_a x$ then $f(x) = a^x$ and $f'(x) = a^x \ln a$

so $$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{a^{\log_a x} \ln a} = \frac{1}{x \ln a}.$$

Ex. Compute the derivative

(a) $\frac{d}{dx} \ln(x^2 + \sin x)$ (b) $\frac{d}{dx} 5^{(\log_3 x + x)}$

(c) $\frac{d}{dx} \frac{\log_5(3x)}{\log_3(5x)}$ (d) $\frac{d}{dx} \log_{10}(e^{5e^x})$

Solution:

(a) $$\frac{d}{dx} \ln(x^2 + \sin x) = \frac{1}{x^2 + \sin x} \cdot (2x + \cos x)$$

(c) $$\frac{d}{dx} \frac{\log_5(3x)}{\log_3(5x)} = \frac{\frac{3}{3x \ln 5} \log_3(5x) - \frac{5}{5x \ln 3} \log_3(5x)}{(\log_3(5x))^2}$$

(b) $$\frac{d}{dx} 5^{(\log_3 x + x)} = 5^{\log_3 x + x} \ln 5 \cdot \left(\frac{1}{x \ln 3} + 1\right)$$

(d) $$\frac{d}{dx} \log_{10}(e^{5e^x}) = \frac{d}{dx}(5e^x \log_{10}(e))$$

$$= 5e^x \log_{10}(e).$$